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An isotropic elastic medium containing a cylindrical borehole with a rigid plug

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Abstract

An analytical and numerical analysis is made of an isotropic elastic medium containing a cylindrical borehole of infinite length in which is located a tightly fitting rigid plug of finite length. Both the pulling of the plug and the occurrence of a radial misfit are considered. The boundary conditions are mixed, with zero radial and shear stresses at the bore surface outside the plug region and displacements given across the plug surface. Using integral representations for a Love auxiliary function, the crucial step is the analytical incorporation of the square root singularity at boundary condition junctions. This is done by using Neumann Bessel function series representations of the integrand kernels of boundary condition stresses such that discontinuous Weber–Schafheitlin integrals can be used to satisfy these conditions exactly. Displacement conditions are solved in terms of integrals of products of Bessel functions. The solutions provide expressions for the far field behaviour of a Kelvin point load solution for the plug pull case and a combined centre of expansion plus double force for radial misfit. Numerical results show good convergence of the method and the correct singular behaviours of borehole surface stresses.

Keywords: Bessel functions; Borehole; Elasticity; Mixed boundary conditions; Singularities

1. Introduction

Practical cases of isotropic media containing cylindrical boreholes, within which are rigid plugs either being withdrawn by bore axial forces or causing radial pressures due to radial misfit, have been examined by Rajapakse and Gross (1996). In both cases, idealised mathematical models to analyse the problem assume the medium to be of infinite extent, axisymmetric in behaviour of displacements and stresses relative to the bore axes, and boundary conditions on the borehole surface which are zero radial and shear stresses outside the plug region and radial and axial displacements at the plug surface.

In these cases of mixed boundary conditions there is a square root singularity for stresses at the boundary condition junctions. This degree of singularity may be proved by the analysis of Zak (1964), justified by the work of Ting et al. (1985), who showed that the dominant behaviour corresponds to that at

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the corner of a two-dimensional wedge in plane strain of opening angle π (Williams, 1952). When examining the method of Rajapakse and Gross (1996), which is one of Fourier integral transforms and solution of resulting multiple integral equations using Green's functions and numerical quadrature, no use is made of the knowledge of this singularity other than to confirm that numerical results suggest this type of behaviour. Here it will be shown that the singular behaviour can be incorporated into the solution process to obtain series solutions in terms of integrals whose coefficients show good numerical convergence. The crucial idea is in the implementation of discontinuous integrals of Weber–Schafheitlin type involving the products of Bessel functions of the first kind. By representing the radial and shear stresses as Fourier cosine and sine integrals and letting their corresponding inversions be represented by Bessel function Neumann series these discontinuous integrals enable the stress conditions to be satisfied identically. The remaining displacement conditions can then be solved readily in terms of Fourier cosine and sine integrals. The same type of procedure has been applied to Laplacean equations in electrostatics by Verolino (1998) and in hydrology by Robinson (2001). Although the work of Sneddon (1966) is extensive in its treatment of mixed boundary value problems and integral equations, the method applied here is not mentioned.

In order to make numerical computations it is necessary to evaluate infinite integrals whose integrands are products of two Bessel functions of the first kind of arbitrary integer orders and an ultimately monotonic function of the integration variable. Because of the, sometimes, complicated oscillatory nature of the integrand and the magnitude of large argument asymptotic behaviour which is algebraic, the accelerated convergence procedures of Lucas (1995) are used.

The following sections define the problem for both plug pull and radial misfit, express axisymmetric displacements and stresses in terms of Love's auxiliary function, and analytically and numerically solve both problems, showing that the remote behaviour for the former is that of the Kelvin solution for a concentrated force and the latter that of a centre of dilatation combined with a double force.

2. Problem formulations

In Fig. 1 are shown local sections of an infinite isotropic elastic medium containing cylindrical boreholes of radii a in which are tightly fitting rigid plugs of lengths $2h$. The axes are the natural ones of z along the bore axes and r radial with origins at the centres of plugs.

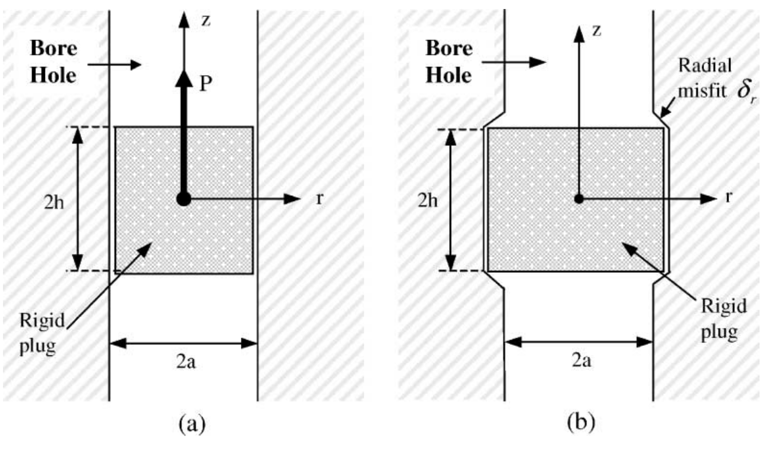

Fig. 1. Schematic for a borehole containing a tightly fitting, rigid plug. (a) Plug pull case, (b) radial misfit.

Fig. 1a represents the plug pull problem with an bore axial force, P. Boundary conditions at $r = a$ in this case are

$$\begin{aligned} \sigma_{rr}(a,z) &= 0, \quad |z| > h \\ \sigma_{rz}(a,z) &= 0, \quad |z| > h \\ u_r(a,z) &= 0, \quad |z| < h \\ u_z(a,z) &= \delta_z, \quad |z| < h \end{aligned} \tag{1}$$

These axisymmetric conditions require that radial stress $\sigma_{rr}(r,z)$ and displacement $u_r(r,z)$ are antisymmetrical with respect to z, and shear stress $\sigma_{rz}(r,z)$ and displacement $u_z(r,z)$ are symmetrical with respect to z.

A relationship connecting P and stress σ_{rz}, and ultimately δ_z, is

$$P = -2\pi a \int_{-h}^{h} \sigma_{rz}(a,z)\,\mathrm{d}z \tag{2}$$

The negative sign is introduced because $\sigma_{rz}(a,z)$ is negative along the contact surface.

The radial misfit problem is depicted in Fig. 1b with boundary conditions

$$\begin{aligned} \sigma_{rr}(a,z) &= 0, \quad |z| > h \\ \sigma_{rz}(a,z) &= 0, \quad |z| > h \\ u_r(a,z) &= \delta_r, \quad |z| < h \\ u_z(a,z) &= 0, \quad |z| < h \end{aligned} \tag{3}$$

Here radial stress $\sigma_{rr}(r,z)$ and displacement $u_r(r,z)$ are symmetrical with respect to z, and shear stress $\sigma_{rz}(r,z)$ and displacement $u_z(r,z)$ are antisymmetrical with respect to z.

Average uniform radial pressure, Q, may be expressed as

$$Q = \frac{1}{2h} \int_{-h}^{h} \sigma_{rr}(a,z)\,\mathrm{d}z \tag{4}$$

and is also ultimately proportional to δ_r.

3. Analytical solutions for plug pull

Axisymmetric displacements and stresses may be defined in terms of Love's auxiliary function, ϕ, in cylindrical coordinates (Love, 1944; Mindlin, 1936; Timoshenko and Goodier, 1951), as

$$\begin{aligned} u_r &= -\frac{1}{2\mu}\frac{\partial^2\phi}{\partial r\partial z} \\ u_z &= \frac{1}{2\mu}\left[2(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\phi \\ \sigma_{rr} &= \frac{\partial}{\partial z}\left[\nu\nabla^2 - \frac{\partial^2}{\partial r^2}\right]\phi \\ \sigma_{\theta\theta} &= \frac{\partial}{\partial z}\left[\nu\nabla^2 - \frac{1}{r}\frac{\partial}{\partial r}\right]\phi \\ \sigma_{zz} &= \frac{\partial}{\partial z}\left[(2-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\phi \\ \sigma_{rz} &= \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\phi \end{aligned} \tag{5}$$

and $u_\theta = 0 = \sigma_{r\theta} = \sigma_{z\theta}$. Elastic constants are Poisson's ratio v and shear modulus $\mu = E/(1+2v)$ with E as Young's modulus. $\nabla^2 = \partial^2/\partial r^2 + (1/r)\partial/\partial r + \partial^2/\partial z^2$ and ϕ satisfies the biharmonic equation

$$\nabla^4\phi = 0 \tag{6}$$

For the Fourier sine transform pair involving the antisymmetrical $\sigma_{rr}(a,z)$, define

$$\begin{aligned} G(\beta) &= \frac{2}{\pi}\int_0^\infty \sigma_{rr}(a,z)\sin\beta z\,\mathrm{d}z = \frac{2}{\pi}\int_0^h \sigma_{rr}(a,z)\sin\beta z\,\mathrm{d}z \\ \sigma_{rr}(a,z) &= \int_0^\infty G(\beta)\sin\beta z\,\mathrm{d}\beta \end{aligned} \tag{7}$$

Similarly for the symmetrical $\sigma_{rz}(a,z)$, define the Fourier cosine transform pair

$$\begin{aligned} H(\beta) &= \frac{2}{\pi}\int_0^\infty \sigma_{rz}(a,z)\cos\beta z\,\mathrm{d}z = \frac{2}{\pi}\int_0^h \sigma_{rz}(a,z)\cos\beta z\,\mathrm{d}z \\ \sigma_{rz}(a,z) &= \int_0^\infty H(\beta)\cos\beta z\,\mathrm{d}\beta \end{aligned} \tag{8}$$

An auxiliary function ϕ which is symmetrical in z, satisfies the biharmonic equation (6), will produce zero displacements and stresses at large r, and can produce the stress representations in (7) and (8) according to (5) is

$$\phi = \int_0^\infty [A(\beta)K_0(\beta r) + B(\beta)\beta r K_1(\beta r)]\cos\beta z\,\mathrm{d}\beta \tag{9}$$

Here $A(\beta)$ and $B(\beta)$ are to be determined and $K_0(\,)$ and $K_1(\,) = -K_0'(\,)$ are modified Bessel functions of the second kind of respective orders 0 and 1 (Watson, 1944). Derivatives involving $\beta r K_1(\beta r)$ which are useful for evaluating displacements and stresses in (5) are $\mathrm{d}/\mathrm{d}r[\beta r K_1(\beta r)] = -\beta^2 r K_0(\beta r)$ and $\nabla^2[\beta r K_1(\beta r)\cos\beta z] = -2\beta^2 K_0(\beta r)\cos\beta z$. The integral representation of ϕ, but not its derivatives in displacements and stresses, may be formal because of singular behaviour at $\beta = 0$. However, this is overcome easily as shown below.

Carrying out the derivatives of ϕ to create $\sigma_{rr}(a,z)$ and $\sigma_{rz}(a,z)$, expressions for $A(\beta), B(\beta)$ in terms of $G(\beta), H(\beta)$ are then found as

$$\begin{aligned} A(\beta) &= \alpha_1 G(\beta) + \alpha_2 H(\beta) \\ B(\beta) &= \alpha_3 G(\beta) + \alpha_4 H(\beta) \end{aligned} \tag{10}$$

where

$$\begin{aligned} \alpha_1 &= \frac{a}{\beta^2\Delta}\frac{1}{K_1(\beta a)}[\beta a W(\beta a) - 2(1-v)] \\ \alpha_2 &= -\frac{a}{\beta^2\Delta}\frac{1}{K_1(\beta a)}[(1-2v)W(\beta a) - \beta a] \\ \alpha_3 &= -\frac{a}{\beta^2\Delta}\frac{1}{K_1(\beta a)} \\ \alpha_4 &= -\frac{a}{\beta^2\Delta}\frac{1}{K_1(\beta a)}\left[W(\beta a) + \frac{1}{\beta a}\right] \\ \Delta &= \beta^2 a^2\left[W^2(\beta a) - 1\right] - 2(1-v) \end{aligned} \tag{11}$$

$$W(\beta a) = \frac{K_0(\beta a)}{K_1(\beta a)} \tag{12}$$

With these values of $A(\beta)$ and $B(\beta)$, the boundary conditions for $u_r(a,z)$ and $u_z(a,z)$, using again the derivatives of ϕ in (5), are

$$\begin{aligned} &a\int_0^\infty [M_1(\beta a)G(\beta)+M_2(\beta a)H(\beta)]\sin\beta z\,\mathrm{d}\beta = 0, \quad |z|<h \\ &a\int_0^\infty [M_3(\beta a)G(\beta)+M_4(\beta a)H(\beta)]\cos\beta z\,\mathrm{d}\beta = 2\mu\delta_z, \quad |z|<h \end{aligned} \tag{13}$$

where

$$\begin{aligned} M_1(\beta a) &= \frac{1}{\varDelta}2(1-v) \\ M_2(\beta a) &= \frac{1}{\varDelta}\left[\beta a\left(W^2(\beta a)-1\right)+2(1-v)W(\beta a)\right] \\ M_3(\beta a) &= M_2(\beta a) \\ M_4(\beta a) &= \frac{1}{\varDelta}\frac{1}{\beta a}\left[\beta a\left((3-2v)W^2(\beta a)-1\right)+4(1-v)W(\beta a)\right] \end{aligned} \tag{14}$$

$K_0(\,), K_1(\,)$ are monotonic functions of their arguments, and their ratio in $W(\,)$ is also monotonic, so then are $M_{1,2,3,4}(\,)$.

To begin the process of satisfying the boundary conditions for $\sigma_{rr}(a,z)$ and $\sigma_{rz}(a,z)$ identically, let $G(\beta)$ and $H(\beta)$ be represented by Bessel function Neumann series (Watson, 1944, Section 16; Eswaran, 1990; Verolino, 1998) to satisfy the odd or even β-conditions required by (7) and (8):

$$G(\beta) = \sum_{m=0}^{\infty} G_m J_{2m+1}(\beta h)$$

$$H(\beta) = \sum_{m=0}^{\infty} H_m J_{2m}(\beta h) \tag{15}$$

where $J_m(\,)$ is an mth order Bessel function of the first kind. The representation (15) is justifiable from the work of Eswaran (1990). He showed that the Fourier transforms, $G(\beta)$ and $H(\beta)$, of functions $\sigma_{rr}(a,z)$ and $\sigma_{zz}(a,z)$ which have compact support, being zero for $|z|>h$, can be represented in the Neumann series form $\beta^{-k}\sum_{m=0}^{\infty} S_m J_{m+k}(\beta h)$. S_m are coefficients of expansion and the exponent k lies in the range $-1 \leqslant k \leqslant 0$. To produce square root singularities k is set to zero.

The particular identities which will be used to satisfy the square root singular behaviour at $r=a, z=\pm h$, are

$$\begin{aligned} \int_0^\infty J_{2m+1}(\beta h)\sin\beta z\,\mathrm{d}\beta &= 0, \quad |z|>h \\ &= (-1)^m \frac{T_{2m+1}(z/h)}{\sqrt{h^2-z^2}}, \quad |z|<h \\ \int_0^\infty J_{2m}(\beta h)\cos\beta z\,\mathrm{d}\beta &= 0, \quad |z|>h \\ &= (-1)^m \frac{T_{2m}(z/h)}{\sqrt{h^2-z^2}}, \quad |z|<h \end{aligned} \tag{16}$$

where $T_m(\,)$ is the mth-order Chebyshev polynomial of the first kind.

These identities are particular cases of the Weber–Schafheitlin integrals (Watson, 1944, Section 13.4)

$$\int_0^\infty \beta^{-w} J_s(\beta h) J_t(\beta z)\,\mathrm{d}\beta, \quad s+t>w>-1 \tag{17}$$

discontinuous at $z=h$, where, for integrals in (16), $\sin\beta z = \sqrt{\pi\beta z/2}J_{1/2}(\beta z)$ and $\cos\beta z = \sqrt{\pi\beta z/2}J_{-1/2}(\beta z)$.

With the representations of $G(\beta)$ and $H(\beta)$ of (15) substituted in the integral expressions (7) and (8) for $\sigma_{rr}(a,z)$ and $\sigma_{rz}(a,z)$ and then making use of the identities (16), the stress boundary conditions are automatically satisfied as well as the singular behaviour at $z=\pm h$ being incorporated. (The remaining non-zero stresses, $\sigma_{\theta\theta}(a,z)$ and $\sigma_{zz}(a,z)$, are also seen to be singular when the stress equilibrium equations are invoked.)

The two integral equations (13) resulting from displacements need to be solved for $G(\beta)$ and $H(\beta)$ or equivalently for all G_m and H_m in (15). This is readily achieved by noting that for $|z|\leqslant h$, z may be replaced by $z=h\sin\omega$ and then introducing the Jacobi identities (Watson, 1944, Section 2.22)

$$\begin{aligned}\sin\beta z &= \sin(\beta h\sin\omega) = 2\sum_{n=0}^{\infty} J_{2n+1}(\beta h)\sin((2n+1)\omega)\\ \cos\beta z &= \cos(\beta h\sin\omega) = \sum_{n=0}^{\infty} \epsilon_{2n} J_{2n}(\beta h)\cos(2n\omega)\end{aligned} \tag{18}$$

where ϵ_n is Neumann's constant, taking the value 1 for $n=0$, otherwise 2. With these identities and the series for $G(\beta)$ and $H(\beta)$ of (15) substituted in the two displacement integrals (13), then assuming interchangeability of integrations and summations, and finally equating left and right sides to cosine or sine series terms, two infinite sets of linear equations arise in G_m and H_m, for $n=0,1,2,\ldots,\infty$:

$$\begin{aligned}&\sum_{m=0}^{\infty}[C_{m,n}G_m + D_{m,n}H_m] = 0\\ &\sum_{m=0}^{\infty}[E_{m,n}G_m + F_{m,n}H_m] = 2\mu\delta_z(2-\epsilon_n)\end{aligned} \tag{19}$$

where

$$\begin{aligned}C_{m,n} &= \int_0^\infty M_1(\beta a)J_{2m+1}(\beta h)J_{2n+1}(\beta h)\,\mathrm{d}\beta\\ D_{m,n} &= \int_0^\infty M_2(\beta a)J_{2m}(\beta h)J_{2n+1}(\beta h)\,\mathrm{d}\beta\\ E_{m,n} &= \int_0^\infty M_3(\beta a)J_{2m+1}(\beta h)J_{2n}(\beta h)\,\mathrm{d}\beta\\ F_{m,n} &= \int_0^\infty M_4(\beta a)J_{2m}(\beta h)J_{2n}(\beta h)\,\mathrm{d}\beta\end{aligned} \tag{20}$$

For the determination of the plug pull force, P, the substitution of the $H(\beta)$ series representation of (15) into the Fourier cosine integral for $\sigma_{rz}(a,z)$ of Eq. (8), then into the integration required in (2) for P, interchanging integrals $\int_0^h\int_0^\infty$ (by Fubini's theorem) and analytically integrating with the aid of the identity (Watson, 1944, Section 13.4)

$$\int_0^\infty J_{2m}(u)\frac{\sin u}{u}\,\mathrm{d}u = \frac{\pi}{2}(2-\epsilon_{2m}) \tag{21}$$

produces the simple result

$$P = -2\pi^2 aH_0 \tag{22}$$

From the linear sets of Eqs. (19), H_0 is proportional to δ_z, and, from (22), so is P.

Viewed from afar, it is expected that the disturbance in the elastic media by the plug is equivalent to that of a point load acting at the centre of the plug. This may be verified by looking at the large $R(=\sqrt{r^2+z^2})$ behaviour for the integral expression for ϕ. This is the same as regarding a and h small relative to R, which in turn is the same as forming small argument expansions of $A(\beta)$ and $B(\beta)$ and then explicitly evaluating the integral. With $A(\beta), B(\beta)$ of Eq. (10) expressed in terms of $G(\beta), H(\beta)$ in their series forms (15) and expanding all the contained Bessel functions, straightforward algebra leads to the large R expression for ϕ:

$$\phi \sim \int_0^\infty \left\{ \mathrm{O}(1) G_0 K_0(\beta r)\cos\beta z + \left(\frac{aH_0}{2(1-v)\beta^2} + \mathrm{O}(1)\right)\beta r K_1(\beta r)\cos\beta z \right\} \mathrm{d}\beta, \quad R=\sqrt{r^2+z^2} \tag{23}$$

This integral is formal in the sense that at $\beta = 0$ the integrand is singular of order $1/\beta^2$. A simple way to remove the formality is to subtract $aH_0(\beta r_0 K_1(\beta r_0)\cos\beta z_0)/(2(1-v)\beta^2)$ where $r_0 \neq 0$, z_0, $R_0 = \sqrt{r_0^2+z_0^2}$ are any constant values. The displacements and stresses are not affected by the addition of a constant to ϕ.

By making use of Basset's integral (Watson, 1944, Section 13.21)

$$\int_0^\infty K_0(\beta r)\cos\beta z \, \mathrm{d}\beta = \frac{\pi}{2R} \tag{24}$$

and a ∇^2 integration of it as

$$\int_0^\infty \frac{1}{\beta^2}[\beta r K_1(\beta r)\cos\beta z - \beta r_0 K_1(\beta r_0)\cos\beta z_0]\, \mathrm{d}\beta = -\frac{\pi}{2}(R-R_0) \tag{25}$$

then the dominant behaviour of ϕ (for displacements and stresses) for large R is

$$\phi \sim \frac{P}{8\pi(1-v)} R \tag{26}$$

which is the expected auxiliary function representation for a Kelvin point load (Mindlin, 1936; Timoshenko and Goodier, 1951).

To complete the full representation of displacements and stresses, expressions for displacements and stresses for all r and z from ϕ-derivatives of Eq. (5) in terms of $G(\beta)$ and $H(\beta)$ are as follows

$$\begin{aligned}
u_r(r,z) &= \frac{a}{2\mu}\int_0^\infty [M_1(\beta r)G(\beta) + M_2(\beta r)H(\beta)]\sin\beta z \, \mathrm{d}\beta \\
u_z(r,z) &= \frac{a}{2\mu}\int_0^\infty [M_3(\beta r)G(\beta) + M_4(\beta r)H(\beta)]\cos\beta z \, \mathrm{d}\beta \\
\sigma_{rr}(r,z) &= \int_0^\infty [M_5(\beta r)G(\beta) + M_6(\beta r)H(\beta)]\sin\beta z \, \mathrm{d}\beta \\
\sigma_{rz}(r,z) &= \int_0^\infty [M_7(\beta r)G(\beta) + M_8(\beta r)H(\beta)]\cos\beta z \, \mathrm{d}\beta \\
\sigma_{\theta\theta}(r,z) &= \int_0^\infty [M_9(\beta r)G(\beta) + M_{10}(\beta r)H(\beta)]\sin\beta z \, \mathrm{d}\beta \\
\sigma_{zz}(r,z) &= \int_0^\infty [M_{11}(\beta r)G(\beta) + M_{12}(\beta r)H(\beta)]\sin\beta z \, \mathrm{d}\beta
\end{aligned} \tag{27}$$

The $M_{1\text{–}12}(\beta r)$ are

$$M_1(\beta r) = \left(\frac{r}{a}\right) F(\beta r)\left\{2(1-v) - \beta a W(\beta a)\left[1 - \left(\frac{r}{a}\right) Z(\beta r)\right]\right\}$$

$$M_2(\beta r) = \left(\frac{r}{a}\right)F(\beta r)\left\{\beta a\left[\left(\frac{r}{a}\right)Z(\beta r)W^2(\beta a) - 1\right] + W(\beta a)\left[1 - 2v + \left(\frac{r}{a}\right)Z(\beta r)\right]\right\}$$

$$M_3(\beta r) = \left(\frac{r}{a}\right)F(\beta r)\left\{\beta a\left[Z(\beta r)W^2(\beta a) - \left(\frac{r}{a}\right)\right] + 2(1-v)Z(\beta r)W(\beta a)\right\}$$

$$M_4(\beta r) = \left(\frac{r}{a}\right)F(\beta r)\left\{(3-2v)Z(\beta r)W^2(\beta a) - \left(\frac{r}{a}\right) + 4(1-v)Z(\beta r)W(\beta a) + \beta^2 a^2 W(\beta a)\left[Z(\beta r) - \left(\frac{r}{a}\right)\right]\right\}$$

$$M_5(\beta r) = F(\beta r)\left\{\beta^2 a^2\left(\frac{r}{a}\right)^2\left[\left(\frac{a}{r}\right)Z(\beta r)W^2(\beta a) - 1\right] + \beta a W(\beta a)\left[1 - \left(\frac{r}{a}\right)Z(\beta r)\right] - 2(1-v)\right\}$$

$$M_6(\beta r) = F(\beta r)\left\{(1-2v)W(\beta a)\left[\left(\frac{r}{a}\right)Z(\beta r) - 1\right] - \beta^2 a^2\left(\frac{r}{a}\right)^2 W(\beta a)\left[1 - \left(\frac{a}{r}\right)Z(\beta r)\right] + \beta a\left(1 - \left(\frac{r}{a}\right)^2\right)\right\}$$

$$M_7(\beta r) = F(\beta r)\left\{\beta^2 a^2\left(\frac{r}{a}\right)W(\beta a)\left[\left(\frac{r}{a}\right)Z(\beta r) - 1\right]\right\}$$

$$M_8(\beta r) = F(\beta r)\left\{\beta^2 a^2\left(\frac{r}{a}\right)^2\left[\left(\frac{a}{r}\right)Z(\beta r)W^2(\beta a) - \left(\frac{r}{a}\right)\right] - \left(\frac{r}{a}\right)\beta a W(\beta a) - \left[1 - \left(\frac{r}{a}\right)Z(\beta r)\right] - 2(1-v)\left(\frac{r}{a}\right)\right\}$$

$$M_9(\beta r) = F(\beta r)\left\{2(1-v) - \beta a W(\beta a)\left[1 - (1-2v)\left(\frac{r}{a}\right)Z(\beta r)\right]\right\}$$

$$M_{10}(\beta r) = F(\beta r)\left\{(1-2v)W(\beta a)\left[1 + \left(\frac{r}{a}\right)Z(\beta r)\right] - \beta a W(\beta a)\left[1 - (1-2v)\left(\frac{r}{a}\right)Z(\beta r)W^2(\beta a)\right]\right\}$$

$$M_{11}(\beta r) = \beta a F(\beta r)\left\{\beta a\left[\left(\frac{r}{a}\right) - Z(\beta r)W^2(\beta a)\right] + 2Z(\beta r)W(\beta a)\right\}$$

$$M_{12}(\beta r) = \beta a F(\beta r)\left\{\left(\frac{r}{a}\right) - 3Z(\beta r)W^2(\beta a) - 2(2-v)\frac{1}{\beta a}Z(\beta r)W(\beta a) + \beta a W(\beta a)\left[\left(\frac{r}{a}\right) - Z(\beta r)\right]\right\}$$

$$F(\beta r) = \frac{a}{r}\frac{1}{\Delta}\frac{K_1(\beta r)}{K_1(\beta a)}, \quad W(\beta r) = \frac{K_0(\beta r)}{K_1(\beta r)}, \quad Z(\beta r) = \frac{W(\beta r)}{W(\beta a)} \tag{28}$$

When $r = a$, $F(\beta a) = 1/\Delta$, $Z(\beta a) = 1$ and the expressions for $M_{1,2,3,4}(\beta a)$ reduce to those already given in (14) as well as $M_5(\beta a) = 1$, $M_6(\beta a) = 0$, $M_7(\beta a) = 0$, $M_8(\beta a) = 1$, consistent with the original definitions of $\sigma_{rr}(a,z)$ and $\sigma_{rz}(a,z)$. The remaining $M_{9\text{–}12}(\beta a)$ are as follows

$$\begin{aligned} M_9(\beta a) &= \frac{1}{\Delta}\{2(1-v) - 2v\beta a W(\beta a)\} \\ M_{10}(\beta a) &= \frac{1}{\Delta}\{\beta a[1-(1-2v)W^2(\beta a)] - 2(1-2v)W(\beta a)\} \\ M_{11}(\beta a) &= \frac{\beta a}{\Delta}\{\beta a[1-W^2(\beta a)] - 2W(\beta a)\} \\ M_{12}(\beta a) &= \frac{\beta a}{\Delta}\left\{1 - 3W^2(\beta a) - 2(2-v)\frac{1}{\beta a}W(\beta a)\right\} \end{aligned} \tag{29}$$

The asymptotic expansion of these $M_{9\text{–}12}(\beta a)$ determines the singular nature of $\sigma_{\theta\theta}(a,z)$ and $\sigma_{zz}(a,z)$. Thus

$$\begin{aligned} M_9(\beta a) &\sim 2v + \mathrm{O}\left(\frac{1}{\beta a}\right) \\ M_{10}(\beta a) &\sim -2v + \mathrm{O}\left(\frac{1}{\beta a}\right) \\ M_{11}(\beta a) &\sim 1 + \mathrm{O}\left(\frac{1}{\beta a}\right) \\ M_{12}(\beta a) &\sim 2 + \mathrm{O}\left(\frac{1}{\beta a}\right) \end{aligned} \tag{30}$$

Further integral identities (Watson, 1944, Section 13.4) required, which are a counterpart to Eqs. (16), are

$$\begin{aligned} \int_0^\infty J_{2m}(\beta h)\sin\beta z\,\mathrm{d}\beta &= \frac{(-1)^m h^{2m}}{\sqrt{z^2-h^2}[z+\sqrt{z^2-h^2}]^{2m}}, \quad |z|>h \\ &= \frac{1}{\sqrt{h^2-z^2}}\sin[2m\sin^{-1}(z/h)], \quad |z|<h \\ \int_0^\infty J_{2m+1}(\beta h)\cos\beta z\,\mathrm{d}\beta &= \frac{(-1)^{m+1} h^{2m+1}}{\sqrt{z^2-h^2}[z+\sqrt{z^2-h^2}]^{2m+1}}, \quad |z|>h \\ &= \frac{1}{\sqrt{h^2-z^2}}\cos[(2m+1)\sin^{-1}(z/h)], \quad |z|<h \end{aligned} \tag{31}$$

On evaluating integrals for $\sigma_{\theta\theta}(a,z)$ and $\sigma_{zz}(a,z)$ of Eqs. (27), with the asymptotic leading terms of $M_{9\text{–}12}(\beta a)$ of (30) and replacing $G(\beta)$ and $H(\beta)$ by their infinite summations, it is clear from the first of identities (16) and (31) that the expected square root singularities are approached from both sides of $|z| = h$. The second of the identities (31) is used for the radial misfit case.

4. Analytical solutions for radial misfit

The solution procedure for radial misfit of the plug is essentially the same as for the plug pull, the differences arising from the reversed symmetry of displacements and stresses. The presentation is thereby briefer.

For $\sigma_{rr}(a,z)$, define the Fourier cosine transform pair

$$\begin{aligned} G(\beta) &= \frac{2}{\pi}\int_0^\infty \sigma_{rr}(a,z)\cos\beta z\,\mathrm{d}z = \frac{2}{\pi}\int_0^h \sigma_{rr}(a,z)\cos\beta z\,\mathrm{d}z \\ \sigma_{rr}(a,z) &= \int_0^\infty G(\beta)\cos\beta z\,\mathrm{d}\beta \end{aligned} \tag{32}$$

Similarly for $\sigma_{rz}(a,z)$, define the Fourier cosine transform pair

$$\begin{aligned} H(\beta) &= \frac{2}{\pi}\int_0^\infty \sigma_{rz}(a,z)\sin\beta z\,\mathrm{d}z = \frac{2}{\pi}\int_0^h \sigma_{rz}(a,z)\sin\beta z\,\mathrm{d}z \\ \sigma_{rz}(a,z) &= \int_0^\infty H(\beta)\sin\beta z\,\mathrm{d}\beta \end{aligned} \tag{33}$$

An appropriate auxiliary function ϕ, antisymmetrical in z, is

$$\phi = \int_0^\infty [A(\beta)K_0(\beta r) + B(\beta)\beta r K_1(\beta r)]\sin\beta z\,\mathrm{d}\beta \tag{34}$$

$A(\beta)$ and $B(\beta)$ again follow from equating ϕ-derived expressions for $\sigma_{rr}(a,z)$ and $\sigma_{rz}(a,z)$ and their representations above:

$$\begin{aligned} A(\beta) &= -\alpha_1 G(\beta) + \alpha_2 H(\beta) \\ B(\beta) &= -\alpha_3 G(\beta) + \alpha_4 H(\beta) \end{aligned} \tag{35}$$

where $\alpha_{1,2,3,4}$ are those for the plug pull problem in Eq. (12).

The boundary conditions for ϕ-derived $u_r(a,z)$ and $u_z(a,z)$, with the expressions above for $A(\beta)$ and $B(\beta)$, produce

$$\begin{aligned} &a\int_0^\infty [M_1(\beta a)G(\beta) - M_2(\beta a)H(\beta)]\cos\beta z\,\mathrm{d}\beta = 2\mu\delta_r, \quad |z| < h \\ &a\int_0^\infty [-M_3(\beta a)G(\beta) + M_4(\beta a)H(\beta)]\sin\beta z\,\mathrm{d}\beta = 0, \quad |z| < h \end{aligned} \tag{36}$$

Again, $M_{1,2,3,4}$ are those of Eq. (14).

Now introduce the Bessel function Neumann series for $G(\beta)$ and $H(\beta)$

$$\begin{aligned} G(\beta) &= \sum_{m=0}^\infty G_m J_{2m}(\beta h) \\ H(\beta) &= \sum_{m=0}^\infty H_m J_{2m+1}(\beta h) \end{aligned} \tag{37}$$

With the representations of $G(\beta)$ and $H(\beta)$ of (37) substituted in the integral expressions (32) and (33) for $\sigma_{rr}(a,z)$ and $\sigma_{rz}(a,z)$ and then making use of the identities (16), the stress boundary conditions are automatically satisfied and produce singular behaviour at $z = \pm h$.

The two integral equations (36) resulting from displacements need to be solved for $G(\beta)$ and $H(\beta)$ or equivalently for all G_m and H_m in (37). With the Jacobi identities (18) and the series for $G(\beta)$ and $H(\beta)$ of (37) substituted in the two displacement integrals (36) producing Fourier series, the two infinite sets of linear equations for G_m and H_m, for $n = 0, 1, 2, \ldots, \infty$ are

$$\begin{aligned} &\sum_{m=0}^\infty [C_{m,n}G_m + D_{m,n}H_m] = 2\mu\delta_r(2 - \epsilon_n) \\ &\sum_{m=0}^\infty [E_{m,n}G_m + F_{m,n}H_m] = 0 \end{aligned} \tag{38}$$

where

$$\begin{aligned} C_{m,n} &= \int_0^\infty M_1(\beta a)J_{2m}(\beta h)J_{2n}(\beta h)\,\mathrm{d}\beta \\ D_{m,n} &= -\int_0^\infty M_2(\beta a)J_{2m+1}(\beta h)J_{2n}(\beta h)\,\mathrm{d}\beta \\ E_{m,n} &= -\int_0^\infty M_3(\beta a)J_{2m}(\beta h)J_{2n+1}(\beta h)\,\mathrm{d}\beta \\ F_{m,n} &= \int_0^\infty M_4(\beta a)J_{2m+1}(\beta h)J_{2n+1}(\beta h)\,\mathrm{d}\beta \end{aligned} \tag{39}$$

For the determination of the average pressure, Q, the substitution of the $G(\beta)$ series representation of (37) into the Fourier cosine integral for $\sigma_{rr}(a,z)$ of Eq. (33), then into the integration required in (4) for Q, and using identity (21),

$$Q = \frac{\pi}{2h}G_0 \tag{40}$$

From the linear sets of Eqs. (38), G_0 is proportional to δ_r, and, from (40), so is Q. All other G_m and H_m are also proportional to δ_r and hence to Q.

The far field behaviour of the auxiliary function ϕ is found in straightforward manner by making small argument expansions of $A(\beta)$ and $B(\beta)$ to produce

$$\phi \sim \int_0^\infty \left\{ \left[a^2 G_0 + \mathrm{O}(\beta^2)\right]K_0(\beta r) - \left[\frac{a^2}{2(1-\nu)}G_0\left(1 - \frac{h}{a}\frac{H_0}{G_0}\right) + \mathrm{O}(\beta^2)\right]\beta r K_1(\beta r)\right\}\frac{\sin\beta z}{\beta}\,\mathrm{d}\beta \tag{41}$$

By using the z integration of Basset's integral (24)

$$\int_0^\infty K_0(\beta r)\frac{\sin\beta z}{\beta}\,\mathrm{d}\beta = \frac{\pi}{4}\log\left(\frac{R+z}{R-z}\right) \tag{42}$$

and the z derivative of Eq. (25)

$$\int_0^\infty \beta r K_1(\beta r)\frac{\sin\beta z}{\beta}\,\mathrm{d}\beta = \frac{\pi}{2}\frac{z}{R} \tag{43}$$

the dominant behaviour of ϕ for large R is

$$\phi \sim \frac{1}{2}a^3\left(\frac{h}{a}\right)Q\left[\log\left(\frac{R+z}{R-z}\right) - \frac{1}{(1-\nu)}\left(1 - \left(\frac{h}{a}\right)\frac{H_0}{G_0}\right)\frac{z}{R}\right] \tag{44}$$

The first term above corresponds to a centre of expansion and the second to an axial double force (Mindlin, 1936; Timoshenko and Goodier, 1951).

The complete set of displacements and stress expressions for arbitrary r and z follow immediately from the plug pull expressions (27)–(30) by noting the following changes when passing from the plug pull to radial misfit cases:

$$\begin{aligned} &\{\sin,\cos\} \to \{\cos,\sin\} \\ &\{M_1,M_2,M_3,M_4,M_5,M_6,M_7,M_8,M_9,M_{10},M_{11},M_{12}\} \\ &\quad \to \{M_1,-M_2,-M_3,M_4,M_5,-M_6,-M_7,M_8,M_9,-M_{10},M_{11},-M_{12}\} \end{aligned} \tag{45}$$

5. Numerical results

With a change of variable from βh to u, the integral expressions for all $C_{m,n}, D_{m,n}, E_{m,n}, F_{m,n}$ of Eqs. (20) and (39) take the form

$$p\int_0^\infty M_i(pu)J_b(u)J_c(u)\,\mathrm{d}u \tag{46}$$

where $i = 1, 2, 3, 4$; $p = a/h$ is the aspect ratio and b and c are positive integers or zero.

In order to evaluate these integrals, their asymptotic behaviours for large u are examined. Firstly an asymptotic expansion for $W(u)$ is

$$W(u) = 1 - \frac{1}{2}\frac{1}{u} + \frac{3}{8}\frac{1}{u^2} - \frac{3}{8}\frac{1}{u^3} + \frac{63}{128}\frac{1}{u^4} - \frac{27}{32}\frac{1}{u^5} + \frac{1899}{1024}\frac{1}{u^6} - \frac{81}{16}\frac{1}{u^7} + \mathrm{O}\left(\frac{1}{u^8}\right) \tag{47}$$

suitable for calculating $W(u)$ for $u > 100$ for 14 significant figure accuracy. Single term asymptotic behaviours for $M_{1,2,3,4}(u)$ then are found to be

$$\begin{aligned} M_1(u) &\sim -2(1-v)\frac{1}{u} + \mathrm{O}\left(\frac{1}{u^2}\right) \\ M_2(u) &\sim -(1-2v)\frac{1}{u} + \mathrm{O}\left(\frac{1}{u^2}\right) \\ M_3(u) &\sim -2(1-v)\frac{1}{u} + \mathrm{O}\left(\frac{1}{u^2}\right) \\ M_4(u) &\sim -(1-2v)\frac{1}{u} + \mathrm{O}\left(\frac{1}{u^2}\right) \end{aligned} \tag{48}$$

i.e., $M_{1,2,3,4} \sim s/u$, where s takes the values $-2(1-v)$ or $-(1-2v)$. Because $M_i(u)$ are monotonic with large argument behaviour defined by (48), all integrals (46) exist, based on the identities (Watson, 1944, Section 13.41,3)

$$I_{bc} = \int_0^\infty \frac{1}{u}J_b(u)J_c(u)\,\mathrm{d}u = \frac{2}{\pi}\frac{\sin(b-c)\pi/2}{b^2-c^2}, \quad b+c \neq 0 \tag{49}$$

$$I_{00} = \int_0^\infty \frac{p^2u}{p^2u^2+1}J_0^2(u)\,\mathrm{d}u = I_0(1/p)K_0(1/p) \tag{50}$$

and $I_0(\,)$ is the zero-order modified Bessel function of the first kind.

Analytical integrations of expressions (46) do not appear to be possible. Moreover, direct numerical quadrature is difficult to do accurately because of the large argument oscillatory behaviour and modest decay of the $J_b(u)$ and $J_c(u)$ Bessel functions with asymptotic property $J_b(u) \sim 1/\sqrt{\pi u}\cos(u - b\pi/2 - \pi/4)$. Some alleviation of this difficulty can be achieved by using the identities (49) and (50) to modify the original integrations of (46):

$$\begin{aligned} &p\int_0^\infty \left[M_i(pu) - \frac{s}{u}\right]J_b(u)J_c(u)\,\mathrm{d}u + psI_{bc}, \quad b+c \neq 0 \\ &p\int_0^\infty \left[M_i(pu) - \frac{spu}{p^2u^2+1}\right]J_0^2(u)\,\mathrm{d}u + sI_{00} \end{aligned} \tag{51}$$

However, obtaining more than three decimal place accuracy is hard to achieve, particularly as the orders b and c increase.

Table 1
Plug pull $P^* = P/\mu a\delta_z$ with Poisson's ratio $\nu = 0.3$ for various aspect ratios, $p = a/h$, compared with the approximate results of Rajapakse and Gross (1996)

p	8	4	2	1	0.8	0.4	0.2
Rajapakse and Gross (1996)	7.73	9.47	11.97		17.92	24.22	37.35
Present P^*	7.668	9.335	11.761	15.426	16.993	23.659	34.505

Table 2
Coefficients $G_{0-3}/\mu a\delta$, $H_{0-3}/\mu a\delta$ with Poisson's ratio $\nu = 0.3$ and $p = a/h = 1$

	G_0	G_1	G_2	G_3	H_0	H_1	H_2	H_3
Plug pull $\delta = \delta_z$	0.20133	−0.07596	0.04525	−0.03193	−0.78151	−0.10675	0.03119	−0.02118
Radial misfit $\delta = \delta_r$	−1.88890	−0.50575	0.08106	−0.04704	−0.40954	0.14987	−0.08985	0.06372

Another approach is to integrate numerically accurately up to a finite $u = u_f$ and then to analytically integrate complete integrand asymptotic expansions (including the $W(pu)$ defined by (47)) in terms of cosine and sine integrals. u_f needs to be determined for each combination of b and c such that the asymptotic expansions are valid.

A more general route, which is used below, is to use the method of Lucas (1995), specifically designed for integrals (46). This recognises that an algebraically damped, single component oscillatory function can be integrated between its zeros to provide an infinite alternating series summable by ε-convergence methods (Sidi, 1988). Then the asymptotic product of $J_b(\,)J_c(\,)$ is proportional to the form $\cos B \cos C$ which is

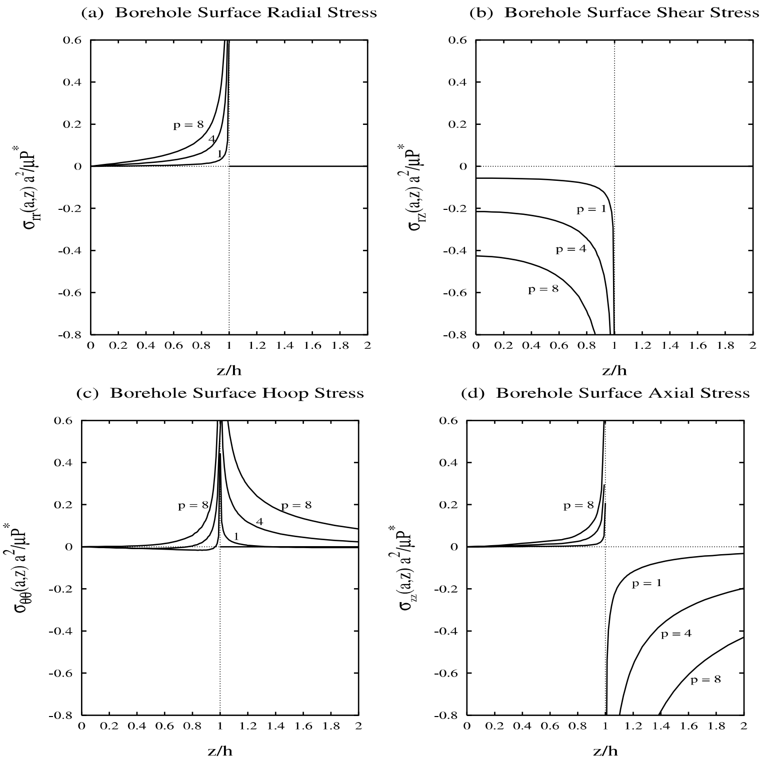

Fig. 2. Plug pull. Borehole surface stresses in plug region, $z/h = 0$–2, aspect ratio $p = a/h = 1, 4, 8$, Poisson's ratio $\nu = 0.3$. (a) Radial stress $\sigma_{rr}(a,z)a^2/\mu P^*$; (b) shear stress $\sigma_{rz}(a,z)a^2/\mu P^*$; (c) hoop stress $\sigma_{\theta\theta}(a,z)a^2/\mu P^*$; (d) axial stress $\sigma_{zz}(a,z)a^2/\mu P^*$.

convertible to the sum of two components $1/2\cos(B+C)$, $1/2\cos(B-C)$. Using the association of Bessel functions of the second kind $Y_b(\,)Y_c(\,)$ at large argument with $\sin B \sin C$, then $J_b(\,)J_c(\,)$ can be replaced by the sum of the two components $1/2[J_b(\,)J_c(\,)+Y_b(\,)Y_c(\,)]$ and $1/2[J_b(\,)J_c(\,)-Y_b(\,)Y_c(\,)]$. The infinite values of $Y_{b,c}(0)$ are avoided by integrating the original integrand up to a finite u_f determined by the zeros of the two components and then summing the alternating series for the two components arising from integrals starting at u_f. The monotonic behaviour of $M_i(pu)$ does not alter the oscillatory behaviour of $J_b(\,)J_c(\,)$ and the method still applies. However, large u values will require the asymptotic expansion of $W(pu)$ which are components of $M_i(pu)$.

The infinite systems of Eqs. (19) and (38) for determining G_m and H_m are truncated at $m = n = N$ to provide a system of $2N+2$ equations under the assumption that increasing N will provide numerical convergence. This is found to be so.

The convergent results for plug pull $P^* = P/\mu a\delta_z$, evaluated from Eq. (22) with $v = 0.3$ for various p are shown in Table 1, where a comparison with the approximate results of Rajapakse and Gross (1996) is also given. The first few significant figures are obtained with N relatively small but the convergence for later

Table 3
Average radial stress $Q^* = Q/\mu a\delta_r$ with Poisson's ratio $v = 0.3$ for various aspect ratios, $p = a/h$, in the radial misfit case

p	8	4	2	1	0.8	0.4	0.2
Q^*	−7.716	−5.219	−3.784	−2.967	−2.790	−2.415	−2.214

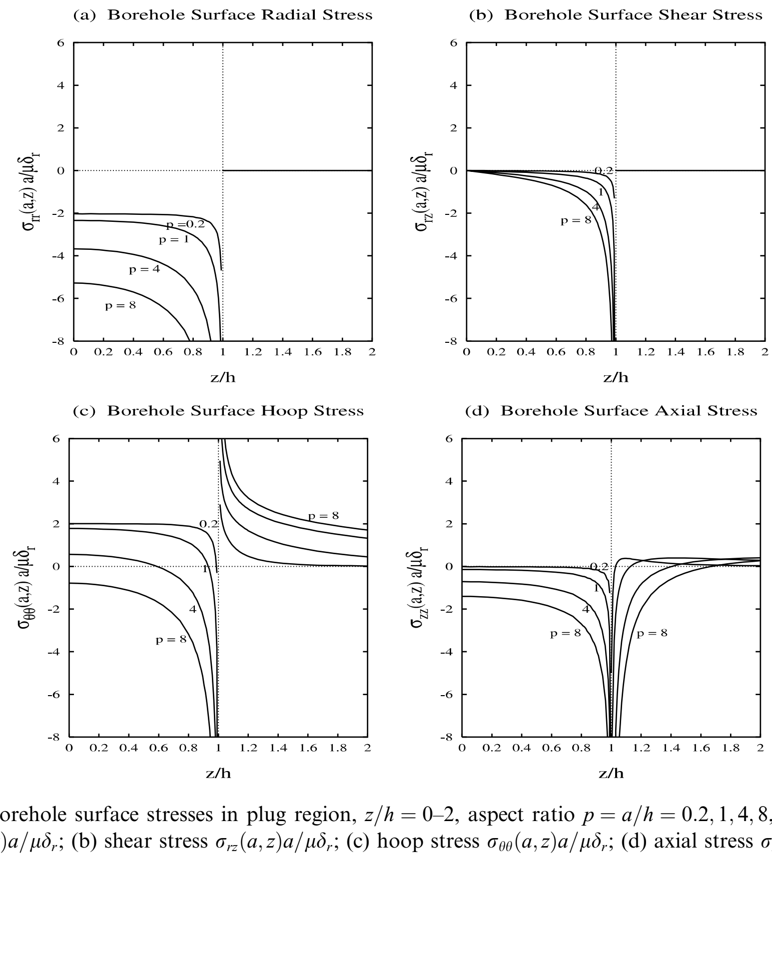

Fig. 3. Radial misfit. Borehole surface stresses in plug region, $z/h = 0$–2, aspect ratio $p = a/h = 0.2, 1, 4, 8$, Poisson's ratio $v = 0.3$. (a) Radial stress $\sigma_{rr}(a,z)a/\mu\delta_r$; (b) shear stress $\sigma_{rz}(a,z)a/\mu\delta_r$; (c) hoop stress $\sigma_{\theta\theta}(a,z)a/\mu\delta_r$; (d) axial stress $\sigma_{zz}(a,z)a/\mu\delta_r$.

figures is slower. For example, at $p = 1$ and $N = [0, 1, 2, 4, 8, 16]$ the corresponding values of P^* are $[15.3626, 15.4182, 15.4229, 15.4252, 15.4260, 15.4263]$. Expectedly, for a short plug with high p the convergence is quicker and conversely so for a long plug. For reference, the coefficients $G_{0\text{–}3}$ and $H_{0\text{–}3}$ are given in Table 2 for both the plug pull and radial misfit cases when $\nu = 0.3$ and $p = 1$.

Fig. 2(a–d) show the scaled stresses $[\sigma_{rr}(a,z), \sigma_{rz}(a,z), \sigma_{\theta\theta}(a,z), \sigma_{zz}(a,z)]a^2/P^*, z/h = 0\text{–}2, p = 1, 4, 8$, for the plug pull equations (27) at $r = a$ and the series expansions for $G(\beta)$ and $H(\beta)$ of (15). This requires the Weber–Schafheitlin identities of (21) for $\sigma_{rr}(a,z)$ and $\sigma_{rz}(a,z)$ and integrals of the form $\int_0^\infty M_i(\beta a)J_b(\beta h)\sin\beta z\,dz$ for $\sigma_{\theta\theta}(a,z)$ and $\sigma_{zz}(a,z)$. These integrals are also evaluable with the method of Lucas (1995) by replacing a $Y_c(\,)$ with cosine or sine. The square root behaviour is evident in all figures and on both sides of $z = h$ for $\sigma_{\theta\theta}(a,z)$ and $\sigma_{zz}(a,z)$. Stresses for $r > a$ are more easily evaluated because of the damping factor, $\exp(-\beta(r-a))$ at large β, arising from $F(\beta r)$ and $Z(\beta r)$ of Eqs. (28) which in turn are dependent on $K_0(\beta r)/K_1(\beta a)$ and $K_1(\beta r)/K_1(\beta a)$.

To complete the numerical exemplification, values of $Q^* = Q/\mu a\delta_r$ with $\nu = 0.3$ defined by Eq. (40) for radial misfit are given in Table 3. These values are in reasonable agreement with those read from a graph presented by Rajapakse and Gross (1996). Convergence rates with N are similar to those of the plug pull case, e.g., at $p = 1$ and $N = [0, 1, 2, 4, 8, 16]$ the corresponding values of Q^* are $[-2.8987, -2.9641, -2.9660, -2.9667, -2.9671, -2.9672]$. Stresses $[\sigma_{rr}(a,z), \sigma_{rz}(a,z), \sigma_{\theta\theta}(a,z), \sigma_{zz}(a,z)]a/\mu\delta_r$, $z/h = 0 - 2$, $p = 0.2, 1, 4, 8$, are shown in Fig. 3(a–d). Expected reversals of symmetry in z are seen. As p becomes small (i.e., long plugs), $\sigma_{\theta\theta}(a,z) + \sigma_{rr}(a,z) \to 0$ over much of $|z|/h < 1$ as seen in Fig. 3(a) and (c) as expected when plain strain conditions are approached.

6. Concluding remarks

By using integrals based on Love's auxiliary biharmonic function, which provides satisfaction of isotropic displacement and stress conditions, analytical solutions to the original mixed boundary problems have been found, albeit in terms of infinite series whose coefficients are determined from an infinite system of linear equations.

The main point to be emphasised in this paper is the recognition and incorporation of the square root singularity in this mixed boundary value problem. This is done by using Neumann Bessel function series representations of the kernels of shear and radial stress integral expressions so that the stress boundary conditions are exactly satisfied with the use of discontinuous Weber–Schafheitlin integrals. The coefficients of the series are found from the displacement conditions across the plug surface which produce an infinite set of linear equations each term consisting of an infinite integral of the product of a monotonic function and two Bessel functions. The integrands which are asymptotically oscillatory and algebraically damped are well calculated using a recent method of Lucas (1995). The truncation of the infinite system to low orders is sufficient to give moderately accurate results.

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